Decay of scalar turbulence revisited

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We demonstrate that at long times the rate of passive scalar decay in a turbulent, or simply chaotic, flow is dominated by regions (in real space or in inverse space) where mixing is less efficient. We examine two situations. The first is of a spatially homogeneous stationary turbulent flow with both viscous and inertial scales present. It is shown that at large times scalar fluctuations decay algebraically in time at all spatial scales (particularly in the viscous range, where the velocity is smooth). The second example explains chaotic stationary flow in a disk/pipe. The boundary region of the flow controls the long-time decay, which is algebraic at some transient times, but becomes exponential, with the decay rate dependent on the scalar diffusion coefficient, at longer times.

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Study of advection of a steadily supplied scalar in smooth flow, pioneered by Batchelor [1] for the case of a slowly changing strain and extended by Kraichnan to the opposite limit of short-correlated flow [2], has developed into a universal theory, applicable to any kind of statistics and temporal correlations of the smooth chaotic flow [3, 4]. (See also the reviews [5, 6].) Extension of this theory to the decay problem [7, 8] suggested a fast (exponential) decay of the passive scalar in the case of spatially smooth (that is approximated by linear velocity profiles at relevant scales) and unbounded random flow. (We will call this case the "pure Batchelor" one.) In parallel, laboratory experiments [9, 10, 11, 12] and numerical simulations [13, 14, 15, 16] were conducted to verify the Batchelor-Kraichnan theory. However, a comparison between theory, experiment and numerical simulations was not conclusive. Different experiments and simulations showed different results, inconsistent with each other and with the theory. In this letter we explain a possible source of the discrepancy. Our major point is that the Batchelor-Kraichnan theory does not apply to the late-time stage of the scalar decay measured in the decay laboratory and numerical experiments [9, 10, 12, 13]. This is because the long-time decay is dominated by the regions of the flow, where the mixing is not as efficient as in the pure Batchelor case.

Before constructing a quantitative theory, let us draw a qualitative physical picture. We start from the first example of a multi-scale turbulent flow with the following hierarchy of scales assumed: $L_v \gg \eta \gg r_d$, where L_v is the energy containing scale of the flow, η is the Kolmogorov (viscous) scale and r_d is the diffusive scale of the scalar. It is known that the passive scalar mixing in the inertial range of turbulent flow is slower at larger scales, since the turnover time grows with scale. This explains the algebraic-in-time decay of the scalar for the inertial range of scales, $L_v \gg r \gg \eta$ [17, 18]. If we now take into account that turbulent velocity is smooth at scales smaller than η, it becomes important to understand what happens with the scalar fluctuations at these small scales. We show in the letter that scalar decay is accompanied by both upscale and downscale transport, however, the situation is asymmetric. The decay of scalar fluctuations in time is due to inertial interval physics, while the spatial correlations of the scalar at the smallest scales are mainly controlled by the viscous part of the flow. (Here, the spatial correlations of the scalar are logarithmic, like in the stationary case [4].) An analogous situation is realized if the scalar is advected by a chaotic (spatially smooth) flow in a box or a pipe. Advection of the scalar slows down in the boundary region, while mixing continues to be efficient in the bulk, i.e. far from the boundaries. The result is that the boundary region supplies the scalar into the bulk. Plumes (blobs) of the scalar, injected from the boundary domain into the box center, cascade in a fast way down to r_d where diffusion smears it out. Therefore, the temporal decay of the scalar is mainly controlled by the rate of scalar injection from the boundary layer. This causes an essential slow down of the decay in comparison with the pure Batchelor case, since in this case there are no stagnation (boundary) regions. The decay of the scalar is algebraic during the transient stage, when the scalar is supplied to the bulk from the boundary region which is wider than the diffusive boundary layer. The boundary region width decreases with time. Whenever the boundary layer width becomes of the order of the diffusive layer width the decrease stops and the algebraic temporal decay turns into an exponential one. The exponential decay rate appears to be parametrically smaller than in the pure Batchelor case.

In all the cases discussed, we consider decay of a passive scalar, θ, described by the equation

$$\partial_t \theta + (\boldsymbol{v}\nabla)\theta = \kappa \nabla^2 \theta, \tag{1}$$

where $\boldsymbol{v}$ is the flow velocity and κ is the diffusion coefficient. Our goal is, starting from the equation (1) and assuming some statistical properties of the velocity field $\boldsymbol{v}$, to describe statistics of the passive scalar decay. In

order to understand the qualitative features of the passive scalar decay, we choose (in the spirit of Kraichnan [19]) the simplest possible case of a short-correlated in time velocity, possessing Gaussian statistics. This model enables us to obtain analytically a time evolution of the passive scalar correlation functions. Two situations are discussed. The first is of a statistically homogeneous flow (infinite volume) with a multi-scale velocity which is smooth at small scales, and possesses a scaling behavior at larger scales. The Reynolds number $Re = (L_v/\eta)^{4/3}$ and the Prandtl number $Pr = (\eta/r_d)^2$ are both assumed to be large. The second situation is $2d$ chaotic flow confined within a disk. We also discuss an extension to the case of a more realistic pipe flow. The flow is assumed to be smooth and obeying the standard viscous behavior in the neighborhood of the boundary: the longitudinal and transverse components of the velocities tend to zero quadratically and linearly (respectively) with the distance to the boundary.

We begin with an analysis of the homogeneous unbounded case. The statistics of a short-correlated in time Gaussian velocity is completely characterized by its pair correlation function

$$\langle v_\alpha(t_1,\boldsymbol{r})v_\beta(t_2,0)\rangle = \delta(t_1-t_2)\left[V_0\delta_{\alpha\beta}-\mathcal{K}_{\alpha\beta}(\boldsymbol{r})\right], \quad (2)$$

$$\mathcal{K}_{\alpha\beta} \equiv \frac{r}{2}W'(r)\left(\delta_{\alpha\beta}-\frac{r_\alpha r_\beta}{r^2}\right)+\frac{d-1}{2}W(r)\delta_{\alpha\beta}, \quad (3)$$

where d is dimensionality of space, V_0 is a constant characterizing fluctuations of the velocity at the integral scale, L_v, and $W(r)$ is a function, which determines the scale-dependence of velocity fluctuations. To describe both viscous and inertial ranges, we introduce

$$W = Dr^2/[1+(r/\eta)^\gamma]\,, \quad (4)$$

where D stands for the amplitude of the velocity fluctuations and γ characterizes the velocity scaling in the inertial interval. The Kolmogorov scale η separates the inertial and viscous intervals. The diffusive length is now expressed as $r_d = \sqrt{\kappa/D}$. Measuring time, t, in the units of $1/D$, one replaces D by unity in all the forthcoming formulas. The object we examine here is the simultaneous pair correlation function of the passive scalar, $F(t,\boldsymbol{r}) = \langle\theta(t,\boldsymbol{r})\theta(t,0)\rangle$, which is the simplest non-zero correlation function in the homogeneous case. Under the assumption of isotropy one derives from Eqs. (1-3)

$$-\,r^{1-d}\partial_r\left\{r^{d-1}\left[W(r)+\kappa\right]\partial_r F_\lambda\right\} = \lambda F_\lambda, \quad (5)$$

where $F_\lambda(r)$ is the Laplace transform of $F(t,r)$ with respect to t. The operator on the left-hand side of Eq. (5) is self-adjoint and non-negative (with respect to the measure $\int \mathrm{d}r\, r^{d-1}$ and under condition that $F(r)$ is smooth at the origin and $\partial_r F(r=0)=0$).

We assume that $\int \mathrm{d}\boldsymbol{r}\,\theta = 0$ (the Corrsin invariant is zero), and also, that initially the scalar is correlated at the scale r_0 from the viscous-convective range, $r_d \ll r_0 \ll \eta$, i.e. $F(0,r)$ is $\approx F(0,0)$ at $r < r_0$ and decays fast enough at $r > r_0$. Then some part of the initial evolution of $F(t,r)$ is not distinguishable from the decay in the pure Batchelor case, described by the limit $\eta \to \infty$ in Eq.(4). Let us briefly describe this initial stage of the dacay. The solutions of Eq. (5) are power-like, $F_\lambda \propto r^{\pm\sqrt{d^2/4-\lambda}-d/2}$, outside the diffusive range, $r \gg r_d$. The functions are normalizable only if $\lambda > d^2/4$. Therefore there is a gap in the spectrum ($0 < \lambda < d^2/4$ are forbidden). The existence of this gap guarantees an exponential decay of $F(t,r)$ with time. This feature is, of course, in agreement with the previous analysis of the pure Batchelor case [7, 8]. A complete set of functions at $r \gg r_d$ is $F_\lambda = r^{-d/2}\cos[\alpha_k + k\ln(r/r_d)]$, where k is a positive number related to λ via $\lambda = d^2/4 + k^2$, and α_k are phases, determined by matching at $r \sim r_d$. Taking into account the orthogonality condition for F_λ, $\int \mathrm{d}r\, r^{d-1}F_\lambda(r)F_{\lambda'}(r) = \pi\delta(k-k')/2$, one can express $F(t,r)$ via $F(0,r)$. Analysis of this expression leads to the following picture. The correlation function, $F(t,r)$, does not change at $r < r_- = r_0\exp(-td)$. At the scales larger than r_-, F decays exponentially, $\propto \exp[-td^2/4]$. Note, that r_-, also, marks the position of the maximum of the spatial derivative of $F(t,r)$, $\partial_r F(t,r)$. Therefore, $r_-(t)$ describes the front running downscale (from r_0). The front reaches r_d at $t_d = \ln(r_0/r_d)/d$, so that at $t > t_d$, $F(t,r)$ decays exponentially at all the small scales. Complimentary to the front running downscale, there exists another front running upscale. Indeed, the integral quantity, $\int_0^s \mathrm{d}r\, r^{d-1}F(t,r)$, which determines the overall amount of the scalar at all the scales smaller than s, achieves its maximum around $r_+ = r_0\exp(td)$.

At $t > t_\eta \sim \ln[\eta/r_0]/D$ the pure Batchelor description ceases to be valid and one does need to account for complete multiscale form of eddy-diffusivity function $W(r)$ given by Eq. (4). Our further analysis is devoted to this general case. The spatial decay of the functions F_λ in the multi-scale case is not as steep as in its pure Batchelor one. It is convenient to change to a new field, Φ_λ, $F_\lambda = r^{1-d}\partial_r[r^{d/2}\Phi_\lambda(z)]$, where $z = 2\sqrt{\lambda}r^{\gamma/2}/\gamma$. The general solution of the resulting equation for $\Phi_\lambda(z)$ is a linear combination of the Bessel functions, $\Phi_\lambda = J_{\pm\nu}(z)$, where $\nu = 2\sqrt{d^2/4-\lambda}/\gamma$. If $\lambda < d^2/4$, the positive root for Φ_λ has to be chosen to satisfy the matching conditions at r_d. The eigen function is normalizable for any positive λ, so that even the smallest positive λ are not forbidden. The lack of the gap in the spectrum of the multi-scale model means an algebraic in time decay of $F(t,r)$, i.e. much slower decay than the one found for the Batchelor-Kraichnan model. We present here the expression for the long-time, $t \gg t_\eta$, asymptotic behavior of $F(t,r)$:

$$F(t,r) \propto \begin{cases} \left[td + \ln(r/\eta) + (r/\eta)^\gamma/\gamma\right]^{-1-d/\gamma}, & r \ll r_+; \\ t^{-5/4-d/2\gamma}r^{-d/2+\gamma/4}, & r \gg r_+ \end{cases} \quad (6)$$

Here $r_+ = (\gamma t)^{1/\gamma}$ stands for position of the front running upscale, r_+ lies in the scaling region $r_+ \gg \eta$, where the multi-scale model turns into the so-called Kraichnan model [19], $W \to r^{2-\gamma}$. (Therefore, it is not surprising, that our result (6) is consistent at $r \gtrsim r_+$ with the expression for the pair correlation function derived before for the Kraichnan model [17, 18] in the regime of zero Corrsin invariant.) Note, that when $t - t_\eta$ is yet moderate in value, the small-scale part of the asymptotic expression (6) is correct only at $r \gg R_-(t) = \eta e^{-td}$, where R_- is the position of the front initiated at $t \sim t_\eta$ and running downscale from η. At $t \sim t_\eta + \ln(\eta/r_d)/d$, the downscale front reaches the dissipative scale, and afterwards the expression (6) is correct for any scales $r \gg r_d$. Eq. (6) shows that just like in the pumped case [1, 2, 4] the scalar structure function is logarithmic, $\langle[\theta(t,\boldsymbol{r})-\theta(t,0)]^2\rangle \sim t^{-2-d/\gamma}\ln[r/r_d]$, at $r_d \ll r \ll \eta$ and large times, where, therefore, the time-dependent factor at the logarithm can be interpreted as a scalar flux from the inertial range, $r \gtrsim \eta$, downscale to the viscous range, $r \lesssim \eta$.

To conclude, the existence of the inertial region in velocity affects drastically the spatio-temporal distribution of the scalar in the viscous range (where the velocity can be approximated by linear profiles). The inertial interval, where the scalar is mainly concentrated, serves as a kind of reservoir for the smaller scales. This explains why the decay of scalar correlation function in the viscous is much slower (algebraic) than in the pure Batchelor case (when it would be exponential).

Now we proceed to the situation of spatially bounded flows. Aiming to demonstrate the main qualitative features of the confined geometry, that is strong sensitivity of the scalar decay to the peripheral (close to the boundary) part of the flow, we examine a model case of $2d$ chaotic (i.e. consisting of only few spatial harmonics) flow inside a disk. (It is, actually, clear that the picture of temporal decay and spatial distribution of the scalar, described below, is universal, i.e. it applies to a typical chaotic flow confined in a close box, particularly in $3d$.) Incompressible flow in $2d$ can be characterized by a stream function ψ, then the radial and azimutal components of the velocity are $v_r = -\partial_\varphi\psi/r$ and $v_\varphi = \partial_r\psi$. Our model of ψ is

$$\psi = -\frac{\xi_1}{2}r^2U(r)\sin(2\varphi) + \frac{\xi_2}{2}r^2U(r)\cos(2\varphi)\,, \qquad (7)$$

$$\langle\xi_i(t_1)\xi_j(t_2)\rangle = 2D\delta_{ij}\delta(t_1-t_2)\,, \qquad (8)$$

where $U(r)$ is a function of r, finite at the origin, $U(0) = 1$, and becoming zero, together with its first derivative, at the disk boundary, $r = 1$, and $\xi_{1,2}(t)$ are zero mean short correlated random Gaussian functions. The value of the passive scalar, θ, averaged over the statistics of ξ, $\langle\theta(t,\boldsymbol{r})\rangle$, is the object of our interest here. One examines how the average concentration of the scalar evolves with time at different locations $\mathbf{r}$ within the disk. The short-correlated feature of the velocity field allows a closed description for $\langle\theta\rangle$ in terms of a partial differential equation. Considering the spherically symmetric part of $\langle\theta\rangle$ only (asymptotically, at large times only this *varphi*-independent part remains essential), and passing from $\langle\theta\rangle$ and r to Υ and q, respectively, where $\langle\theta\rangle = r^{-1}\partial_r[r^2\Upsilon(t,q)]$ and $q = -\ln r$, one finds that the Laplace transform of Υ with respect to t (measured, again, in D^{-1} units) satisfies

$$(U^2 + \kappa e^{-2q})(\partial_q^2\Upsilon_\lambda - 2\partial_q\Upsilon_\lambda) + \lambda\Upsilon_\lambda = 0. \qquad (9)$$

The diffusion term is important only at the center of the disk and near the boundaries. Analysis of Eq. (9) is straightforward but bulky. Below we will present only selected details of the analysis, aiming to describe the general picture of the phenomenon. (The complete account for the derivation details will be published elsewhere [20].)

In the bounded flow the decay of passive scalar splits into three distinct stages. The major effect dominating the first stage (just as in the pure Batchelor case, explained by Eq. (9) with $U \to 1$) is formation of elongated structures (stripes) of the scalar in the bulk region of the flow. The stripes are getting thinner with time, i.e. inhomogeneities of smaller and smaller scales are produced. Once the width of the stripes decreases down to the dissipative scale, $r_d \sim \sqrt{\kappa}$, the stripes are smeared out by diffusion. The stretching-contraction process is exponential in time, so that the initial stage when the stripes are formed lasts for $\tau_1 \sim \ln[1/r_d]$. By the end of this first stage the scalar is exhausted in the central region of the flow. The stretching rate, however, is smaller in the peripheral domain than in the bulk. Thus the scalar literally remains longer in the peripheral domain. This defines the second, transient, stage. At $\kappa^{1/4} \ll q \approx 1 - r \ll \lambda^{1/4} \ll 1$, the solution of Eq. (9) is $\Upsilon_\lambda \propto q\sin\left(\alpha_\lambda + \sqrt{\lambda}q/U\right)$. In the other asympotitc region at $q \gg \lambda^{1/4}$ one can drop the second derivative (with respect to q) and diffusive terms in Eq. (9). One finds that, $\partial_q \ln\Upsilon_\lambda \sim 1$ at $q \sim \lambda^{1/4}$. Matching those two asymptotic regions at $q \sim \lambda^{1/4}$ we come to the $\alpha_\lambda = 0$ condition. Once the asymptotic form of the eigenfunction Υ_λ is known, it is straightforward to restore the behavior of $\Upsilon(t,q)$. One finds that $\Upsilon(t,q)$ (and, therefore, $\langle\theta\rangle$) is concentrated in the $\delta(t) \sim 1/\sqrt{t}$-small (and shrinking with time) vicinity of the boundary. In the domain outside of the shrinking layer the decay is algebraic, $\langle\theta\rangle \propto t^{-3/2}q^{-3}$. The second stage lasts for $\tau_2 \sim \kappa^{-1/2}$, i.e. until $\delta(t)$ shrinks down to $r_{bl} = \kappa^{1/4}$, which is the width of the diffusive boundary layer. Account for diffusivity in the $q \ll 1$ analysis gives yet another matching condition for Υ_λ, at $r \sim r_{bl}$, resulting in formation of a discrete spectrum for λ. The smallest λ (and therefore the value of the level spacing in the discrete spectrum) is estimated by $\sqrt{\kappa}$. (Using the periodic character of

the sin-function, one finds that the n-th level eigenvalue λ_n is estimated by $\sim n^2\sqrt{\kappa}$ at large n.) Therefore, at $t \gg \tau_2$, $\langle\theta\rangle$ decays exponentially, $\propto \exp(-t/t_d)$, where $t_d \sim \kappa^{-1/2}$. The smallest λ eigen-function is localized at $q \sim r_{bl}$. Thus the third (final) stage of evolution is characterized by the majority of the scalar remaining in the diffusive boundary layer.

To conclude, the temporal behavior of the passive scalar correlations, predicted by the bounded flow theory, is complicated. It involves two different exponential regimes separated by an algebraic one, so that a special accuracy is required in order to quantify the theory against various experiments in the field.

The bounded flow theory can be applied to the experiment of Groisman and Steinberg [12], where the passive scalar is advected through a pipe by dilute-polymer-solution flow. The polymer-related elastic instability makes the flow chaotic. The mean flow along the pipe is also essential, so that according to the standard Taylor hypothesis, measurements of scalar concentration at different positions along the pipe are interpreted as correspondent to different times in an artificial decay problem. Exponential decay of the scalar was reported in [12] at long times (long pipes). The theoretical picture explaining the experiment is similar to the one proposed above for the disk. The width of the diffusive boundary layer near the pipe boundaries is estimated by $R\cdot Pe^{-1/4}$, where R is the pipe radius and Pe is the Peclet number, $Pe = R^2\sigma/\kappa$ and σ is a typical value of the velocity gradient (Pe is $\sim 10^4$ in the conditions of [12]). The characteristic decay time (fixed by the diffusive boundary layer width) is estimated by $t_d \sim \sigma^{-1}Pe^{1/2}$. However, one should be careful when converting this time to the decrement of the passive scalar decay along the direction of the flow (pipe). The average velocity of the flow tends to zero near the boundary of the pipe, making the advection near the boundary less efficient than in the bulk. Assuming a linear profile of the average velocity near the boundary, one gets the following estimate for the law of scalar fluctuation decay along the pipe, $\sim \exp(-\gamma z/u_0)$, where z is the coordinate in the direction of the pipe, u_0 is the average velocity at the center of the pipe, and $\gamma \sim \sigma\, Pe^{-1/4}$. The factor $Pe^{-1/4}$ here is a manifestation of a slow down of the passive scalar decay in comparison with the pure Batchelor case. The same factor characterizes decay of higher order correlation functions of the scalar as well. For example, one can consider the passive scalar pair correlation function $F(r)$ near the center of the disk. One finds that $F(r) \propto r^{-\alpha}$, where α is small and can be estimated as $\alpha \sim \mathrm{Pe}^{-1/4}$. (Note, that this behavior is almost indistinguishable from the logarithmic one, found for the stationary case [1, 2, 4].) This estimations agree with the experimental data of [12].

Let us now briefly discuss two other available numerical and experimental results. Pierrehumbert [13] reported exponential decay of the scalar correlations in his numerical experiment with chaotic map velocity in a periodic box. This observation is consistent with our results, since the periodic boundary conditions are less restrictive than the zero condition for velocity at the box boundary, and, therefore, should leave a finite gap in the λ-spectrum even in the limit $\kappa \to 0$. In the experiment of Jullien, Castiglione and Tabeling on $2d$ stationary flow steered by magnets in a finite volume beneath the $2d$ layer [11], the decay rate of the scalar seems slower than exponential (Fig. 2 of [11]). This observation is in agreement with the absence of the gap in the λ-spectrum (at $\kappa \to 0$) we found for the finite box model.

The brevity of this letter does not allow us to discuss effects of intermittency (which manifests in high-order moments of the scalar), of higher order angular harmonics (in the disk or pipe geometry), and details of the spatial distribution of the scalar in a variety of other inhomogeneous cases (e.g. periodic flow). This detailed discussion is postponed for a longer paper to be published elsewhere [20] (where we will also present results of direct numerical simulations of the passive scalar decay for these cases).

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[1] G. K. Batchelor, JFM **5**, 113 (1959).
[2] R. H. Kraichnan, Phys. Fluids. **10**, 1417 (1967); JFM **47**, 525 (1971); Ibid **67**, 155 (1975).
[3] B.I. Shraiman, E.D. Siggia, Phys. Rev. E **49**, 2912 (1994).
[4] M. Chertkov, G. Falkovich, I. Kolokolov, and V. Lebedev, Phys. Rev. E **51**, 5609 (1995).
[5] B. I. Shraiman and E. D. Siggia, Nature **405**, 639 (2000).
[6] G. Falkovich, K. Gawędzki, and M. Vergassola, Rev. Mod. Phys. **73**, 913 (2001).
[7] D. T. Son, Phys. Rev. E **59**, R3811 (1999).
[8] E. Balkovsky, A. Fouxon, Phys. Rev. E **60**, 4164 (1999).
[9] P. L. Miller and P. E. Dimotakis, JFM **308**, 129 (1996).
[10] B. S. Williams, D. Marteau, and J. P. Gollub, Phys. Fluids **9**, 2061 (1997).
[11] M.-C. Jullien, P. Castiglione, and P. Tabeling, Phys. Rev. Lett. **85**, 3636 (2000).
[12] A. Groisman and V. Steinberg, Nature **410**, 905 (2001).
[13] R. T. Pierrehumbert, Chaos. Sol. Fract. **4**, 1091 (1993).
[14] M. Holzer and E. Siggia, Phys. Fluids **6**, 1820 (1994).
[15] T. M. Antonsen, F. F. Fan, and E. Ott, Phys. Rev. Lett **75**, 1751 (1995).
[16] D. Bogucki, J. A. Domaradzki, and P. K. Yeung, JFM **343**, 111 (1997).
[17] G. Eyink and J. Xin, J. Stat. Phys. **100**, 679 (2000).
[18] M. Chaves, G. Eyink, U. Frisch, and M. Vergassola, Phys. Rev. Lett. **86**, 2305 (2001).
[19] R. H. Kraichnan, Phys. Rev. Lett. **72**, 1016 (1994).
[20] M. Chertkov, V. Lebedev, and Z. Toroczkai, in preparation.